HECKE ALGEBRA CORRESPONDENCES FOR THE METAPLECTIC GROUP

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Abstract. Over a p-adic field of odd residual characteristic, Gan and Savin proved a correspondence between the Bernstein components of the even and odd Weil representations of the metaplectic group and the components of the trivial representation of the equal rank odd orthogonal groups. In this paper, we extend their result to the case of even residual characteristic.

Introduction

Fix a nonarchimedean local field k of residue characteristic p and characteristic different from 2. Let W be a non-degenerate symplectic space over k of dimension $2n$ and $\widetilde{\mathrm{Sp}}(W)$ the 2-fold metaplectic cover of $\mathrm{Sp}(W)$. For an additive character ψ of k, let ω_ψ be the Weil representation of $\widetilde{\mathrm{Sp}}(W)$, which decomposes into its even and odd constituents, $\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-$. In the category of genuine, smooth representations of $\widetilde{\mathrm{Sp}}(W)$, let $\mathcal{G}_\psi^\pm$ be the component containing $\omega_\psi^\pm$.

Consider quadratic spaces $V^\pm$ of dimension $2n+1$ with trivial discriminant, where V^+ has the trivial Hasse invariant and V^- the non-trivial one. Then, $\mathrm{SO}(V^+)$ is the split adjoint group of type B_n and $\mathrm{SO}(V^-)$ is its unique non-split inner form. In the category of smooth representations of $\mathrm{SO}(V^\pm)$, let $\mathcal{S}_0^\pm$ be the component containing the trivial representation of $\mathrm{SO}(V^\pm)$.

Let ϵ be $+$ or $-$. In [GS2], Gan and Savin proved an equivalence of categories between $\mathcal{G}_\psi^\epsilon$ and $\mathcal{S}_0^\epsilon$ assuming $p \neq 2$. The aim of this paper is to extend their result to the case of even residual characteristic.

We follow their general strategy of exploiting minimal types of the Weil representation to define a Hecke algebra H_ψ^ϵ, showing that the category $\mathcal{G}_\psi^\epsilon$ is equivalent to the category of H_ψ^ϵ-modules, and giving an isomorphism between H_ψ^ϵ and the standard Iwahori-Hecke algebra of $\mathrm{SO}(V^\epsilon)$.

A key ingredient for extending their result is an analysis of the K-types of the Weil representation in arbitrary residual characteristic which was carried out by Savin and the second-named author in [SW]. We also employ the machinery of Bushnell, Henniart, and Kutzko in [BHK] to compare the Plancherel measures induced from the respective Hecke algebras. More explicitly, the layout of the paper is as follows.

§1. We introduce notation and summarize some relevant background material.

§2. We describe a minimal type for an open compact subgroup and compute the corresponding spherical Hecke algebra H_ψ^ϵ. We give an isomorphism between H_ψ^ϵ and the standard Iwahori-Hecke algebra H^ϵ of $\mathrm{SO}(V^\epsilon)$. We show that the isomorphism $H_\psi^\epsilon \cong H^\epsilon$ is, in fact, an isomorphism of Hilbert algebras with involution, thus giving a coincidence of induced Plancherel measures under suitable normalization. A corollary of this result is that the correspondence of Hecke algebra modules preserves formal degree.

§3. We prove that the category of H^ϵ_ψ-modules is equivalent to the category $\mathcal{G}^\epsilon_\psi$, thus giving the desired equivalence of categories $\mathcal{G}^\epsilon_\psi \cong \mathcal{S}^\epsilon_0$. From the theory of Plancherel measures, we deduce that this equivalence preserves the temperedness and square-integrability of representations.

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1. Preliminaries

Throughout the paper, k is a nonarchimedean local field with residual characteristic p; we allow for arbitrary residual characteristic but assume that the characteristic of k is different from 2. Let $\mathcal{O}$ and ϖ be the ring of integers and a chosen uniformizer, respectively. Denote by q the cardinality of the residue field and by e the valuation of 2 in k. If $p = 2$, then e is the ramification index of 2; otherwise $e = 0$. Let ψ be a non-trivial additive character on k; for convenience, we assume that ψ has conductor $2e$, i.e., that $4\mathcal{O}$ is the largest additive subgroup of $\mathcal{O}$ on which ψ acts trivially.

For a vector space V over k, we denote by $S(V)$ the Schwartz space of smooth, compactly supported, $\mathbb{C}$-valued functions on V. We denote the subspaces of even and odd functions in $S(V)$ by $S(V)^+$ and $S(V)^-$, respectively.

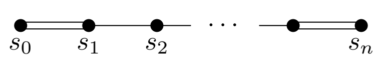
1.1. **The symplectic group** $\mathrm{Sp}(W)$.

Let W be a non-degenerate symplectic space of dimension $2n$ with basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{f}_1, \ldots, \mathbf{f}_n\}$, where $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0 = \langle \mathbf{f}_i, \mathbf{f}_j \rangle$ for all i, j and $\langle \mathbf{e}_i, \mathbf{f}_j \rangle = \delta_{i,j}$. The symplectic group $\mathrm{Sp}(W)$ is the group of invertible transformations of W which are invariant under the symplectic form. The decomposition $W = X + Y$, where X is the span of the $\mathbf{e}_i$ and Y is the span of the $\mathbf{f}_i$ is a polarization of W.

Let $\Sigma = \{\pm\epsilon_i \pm \epsilon_j : 1 \le i < j \le n\} \cup \{\pm 2\epsilon_i : 1 \le i \le n\}$ be the root system of $\mathrm{Sp}(W)$, where ϵ_i has the usual meaning as a character on a maximal torus. We take $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ to be the set of simple roots, where $\alpha_n = 2\epsilon_n$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ otherwise.

Let $\Sigma_a = \{\alpha + m : \alpha \in \Sigma, m \in \mathbb{Z}\}$ be the set of affine roots; the affine action of Σ_a on the maximal torus is given by $(\alpha + m)(h) = \alpha(h) + m$. We take $\Delta_a = \Delta \cup \{\alpha_0\}$ to be the set of simple affine roots, where $\alpha_0 = -2\epsilon_1 + 1$.

For each affine simple root α_i, let s_i be the corresponding affine simple reflection. The Weyl group Ω is the group generated by the simple reflections $s_1, \ldots, s_n$. The affine Weyl group Ω_a is the group generated by the affine simple reflections $s_0, s_1, \ldots, s_n$; it is the semi-direct product $\Omega_a = D\Omega$ of a translation group D and the Weyl group. Both Ω and Ω_a are Coxeter groups whose braid relations are given according to the following Coxeter diagram.



For each root $\alpha \in \Sigma$, we fix a map $\Phi_\alpha : \mathrm{SL}_2(k) \to \mathrm{Sp}(W)$ such that the images of the unipotent upper and lower triangular matrices in $\mathrm{SL}_2(k)$ are the root subgroups of $\mathrm{Sp}(W)$ corresponding to α and $-\alpha$, respectively. For each affine root $\alpha + m \in \Sigma_a$, we define the map $\Phi_{\alpha+m} : \mathrm{SL}_2(k) \to$

$\mathrm{Sp}(W)$ by

$$\Phi_{\alpha+m}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Phi_\alpha \begin{pmatrix} a & \varpi^m b \\ \varpi^{-m} c & d \end{pmatrix};$$

we write

$$\begin{aligned} x_{\alpha+m}(t) &= \Phi_{\alpha+m}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_\alpha(\varpi^m t) \qquad (t \in k), \\ w_{\alpha+m}(t) &= \Phi_{\alpha+m}\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} = w_\alpha(\varpi^m t) \qquad (t \in k^\times), \\ h_{\alpha+m}(t) &= \Phi_{\alpha+m}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = h_\alpha(t) \qquad (t \in k^\times). \end{aligned}$$

We take the element $w_{\alpha_i}(1)$ as a representative in $\mathrm{Sp}(W)$ of the simple affine reflection s_i. We will frequently use the same notation to refer to an element $w = s_{i_1} \cdots s_{i_r}$ in Ω_a and its representative $w = w_{\alpha_{i_1}}(1) \cdots w_{\alpha_{i_r}}(1)$ in $\mathrm{Sp}(W)$.

1.2. **Open compact subgroups of** $\mathrm{Sp}(W)$**.** For $i = 0, \ldots, n$, we define the lattice

$$\mathcal{L}_i = \mathrm{Span}_{\mathcal{O}}\{\mathbf{e}_1, \ldots, \mathbf{e}_n, \varpi\mathbf{f}_1, \ldots, \varpi\mathbf{f}_i, \mathbf{f}_{i+1}, \ldots, \mathbf{f}_n\}.$$

The stabilizer $K_i = \{g \in \mathrm{Sp}(W) : g\mathcal{L}_i \subseteq \mathcal{L}_i\}$ is a maximal open compact subgroup of $\mathrm{Sp}(W)$; it is the group generated by those $\Phi_{\alpha_j}(\mathcal{O})$ for which $j \neq i$. Every maximal open compact subgroup of $\mathrm{Sp}(W)$ is conjugate to one of the K_i.

The intersection of $K_0, \ldots, K_n$ is an Iwahori subgroup I. The double cosets in $I \backslash \mathrm{Sp}(W)/I$ are parameterized by the affine Weyl group; namely, each I-double coset is of the form IwI for some $w \in \Omega_a$. The number of I-single cosets in IwI is

$$[IwI : I] = q^{\ell(w)},$$

where ℓ is the length function on Ω_a.

1.3. **Metaplectic group and the Weil representation.** For a polarization $W = X + Y$, the Schwartz space $S(Y)$ realizes the unique (up to isomorphism) representation ρ_ψ of the Heisenberg group with the central character ψ. Via the action of $\mathrm{Sp}(W)$ on the Heisenberg group, ρ_ψ gives a projective representation of $\mathrm{Sp}(W)$ which lifts to a linear representation ω_ψ, called the Weil representation, of the central extension $\mathrm{Sp}'(W)$ of $\mathrm{Sp}(W)$ given by

$$1 \to \mathbb{C}^\times \to \mathrm{Sp}'(W) \to \mathrm{Sp}(W) \to 1.$$

It is a theorem of Weil that the derived group of $\mathrm{Sp}'(W)$ is the 2-fold cover $\widetilde{\mathrm{Sp}}(W)$ of $\mathrm{Sp}(W)$ and that ω_ψ is a faithful representation of $\widetilde{\mathrm{Sp}}(W)$.

For each subgroup $H \subseteq \mathrm{Sp}(W)$, we denote its preimage in $\widetilde{\mathrm{Sp}}(W)$ by $\widetilde{H}$. For each root α, the element $x_\alpha(t)$ canonically lifts to an element $\widetilde{x}_\alpha(t)$ in $\widetilde{\mathrm{Sp}}(W)$. We may therefore define lifts of $w_\alpha(t)$ and $h_\alpha(t)$ via the formulas

$$\begin{aligned} \widetilde{w}_\alpha(t) &= \widetilde{x}_\alpha(t)\widetilde{x}_{-\alpha}(-t^{-1})\widetilde{x}_\alpha(t), \\ \widetilde{h}_\alpha(t) &= \widetilde{w}_\alpha(t)\widetilde{w}_\alpha(-1). \end{aligned}$$

We will take $\widetilde{w}_{\alpha_i}(1)$ for a representative in $\widetilde{\mathrm{Sp}}(W)$ of the affine simple reflection s_i. We will continue to abuse notation when referring to an element of Ω_a or its representatives in either $\mathrm{Sp}(W)$ or $\widetilde{\mathrm{Sp}}(W)$.

1.4. **Minimal types of the Weil representation.** Realized as a representation of $S(Y)$, the Weil representation ω_ψ decomposes into the sum of even and odd functions, $\omega_\psi^+ \oplus \omega_\psi^-$. We consider the lattices $L_i = \mathcal{L}_i \cap Y$. As computed in [SW], $\widetilde{K}_i$ acts on $\tau_i = S(L_0/2L_i)$, viewed naturally as a subspace of $S(Y)$.

The space τ_0 consists entirely of even functions and is an irreducible $\widetilde{K}_0$-module. Otherwise, as a $\widetilde{K}_i$-module, τ_i decomposes as $\tau_i^+ \oplus \tau_i^-$. Each $\tau_i^\pm$ admits a tensor product structure,

$$\tau_i^\pm = S(\mathcal{O}\mathbf{f}_1/2\varpi\mathcal{O}\mathbf{f}_1)^\pm \otimes \cdots \otimes S(\mathcal{O}\mathbf{f}_i/2\varpi\mathcal{O}\mathbf{f}_i)^\pm \otimes S(\mathcal{O}\mathbf{f}_{i+1}/2\mathcal{O}\mathbf{f}_{i+1}) \otimes \cdots \otimes S(\mathcal{O}\mathbf{f}_n/2\mathcal{O}\mathbf{f}_n),$$

hence the dimension of $\tau_i^\pm$ is $\frac{1}{2}q^{en}(q^i \pm 1)$. Note that $\tau_i \subseteq \tau_{i+1}$ for $0 \le i < n$.

1.5. **Spherical Hecke algebras.** We summarize some generalities on Hecke algebras found in [GS2].

Let G be a totally disconnected topological group and $K \subseteq G$ an open compact subgroup; fix a Haar measure dg on G. For an irreducible, finite-dimensional representation (σ, V_σ) of K, let (σ^*, V_σ^*) be its contragredient representation and define the σ-spherical Hecke algebra by

$$\mathcal{H}(G /\!\!/ H; \sigma) = \left\{ f : G \to \mathrm{End}(V_\sigma^*) : \begin{array}{l} f \text{ is smooth and compactly supported,} \\ f(k_1 g k_2) = \sigma^*(k_1) f(g) \sigma^*(k_2), \text{ for } k_i \in K, g \in G \end{array} \right\};$$

it is an algebra under convolution with an identity element which we denote 1_σ.

For a smooth representation (π, V_π) of G, consider the space $(V_\pi \otimes V_\sigma^*)^K$ of K-fixed vectors in $V_\pi \otimes V_\sigma^*$; this space admits a natural action of $\mathcal{H}(G /\!\!/ K; \sigma)$ by

$$\pi(f)(v \otimes e) = \int_G \pi(g) v \otimes f(g) e \, dg,$$

where $v \otimes e \in (V_\pi \otimes V_\sigma^*)^K$ and $f \in \mathcal{H}(G /\!\!/ K; \sigma)$.

Let Γ be an open compact subgroup of G containing K; assume that the index $[\Gamma : K]$ is finite. We consider $\mathcal{H}(\Gamma /\!\!/ K; \sigma)$ as a finite-dimensional subalgebra of $\mathcal{H}(G /\!\!/ K; \sigma)$ via

$$\mathcal{H}(\Gamma /\!\!/ K; \sigma) = \{ f \in \mathcal{H}(G /\!\!/ K; \sigma) : \mathrm{supp}(f) \subseteq \Gamma \}.$$

We have a natural isomorphism $L : \mathcal{H}(\Gamma /\!\!/ K; \sigma) \xrightarrow{\sim} \mathrm{End}_\Gamma(\mathrm{Ind}_K^\Gamma(\sigma^*))$ given by

$$(L(f)\phi)(g) = \int_\Gamma f(h) \phi(h^{-1} g) \, dh$$

for $f \in \mathcal{H}(\Gamma /\!\!/ K; \sigma)$, $\phi \in \mathrm{Ind}_K^\Gamma(\sigma^*)$, and $g \in \Gamma$.

Suppose that (π, V_π) is an irreducible, smooth, finite dimensional representation of Γ such that $(V_\pi \otimes V_\sigma^*)^K \neq 0$. Then $(V_\pi \otimes V_\sigma^*)^K$ is a simple $\mathcal{H}(\Gamma /\!\!/ K; \sigma)$-module via the action of $\mathcal{H}(G /\!\!/ K; \sigma)$. Now assume $\mathcal{H}(\Gamma /\!\!/ K; \sigma)$ is commutative. Then $(V_\pi \otimes V_\sigma^*)^K$ is one dimensional, and the action of $\mathcal{H}(\Gamma /\!\!/ K; \sigma)$ factors through a maximal ideal $\mathfrak{m} \subseteq \mathcal{H}(\Gamma /\!\!/ K; \sigma)$. Moreover

$$\mathrm{Ind}_K^\Gamma(\sigma^*) / \Big(L(\mathfrak{m}) \cdot \mathrm{Ind}_K^\Gamma(\sigma^*) \Big) \cong \pi^*.$$

Therefore if $(\pi_1, V_1), \ldots, (\pi_l, V_r)$ are the irreducible representations (up to isomorphism) of Γ such that $(V_i \otimes V_\sigma^*)^K \neq 0$, then we have

$$\mathrm{Ind}_K^\Gamma(\sigma^*) \cong \pi_1^* \oplus \cdots \oplus \pi_r^*.$$

For each $f \in \mathcal{H}(\Gamma /\!\!/ K; \sigma)$, the trace of $L(f)$ is $\lambda_1 d_1 + \cdots + \lambda_r d_r$, where $d_i = \dim V_i$ and $\lambda_i = \pi_i(f)$. If f is not supported on K, then the trace of $L(f)$ is 0. The case of $r = 2$ is summarized by the following lemma.

Lemma 1.1. *Suppose that* $\dim \mathcal{H}(\Gamma /\!/ K;\sigma) = 2$ *with* $T \in \mathcal{H}(\Gamma /\!/ K;\sigma)$ *not supported on* K. *Let* (π_i, V_i), *for* $i = 1, 2$, *be the two irreducible representations (up to isomorphism) of* Γ *such that* $(V_i \otimes V_\sigma^*)^K \neq 0$. *Write* $d_i = \dim V_i$ *and* $\lambda_i = \pi_i(T)$. *Then,*

1. $\lambda_1 d_1 + \lambda_2 d_2 = 0$;
2. *the dimension of* $\mathrm{Ind}_K^\Gamma(\sigma^*)$ *is* $d = d_1 + d_2$;
3. *the minimal polynomial of* T *is* $(T - \lambda_1)(T - \lambda_2) = 0$.

There is additional structure on $\mathcal{H}(G /\!/ K;\sigma)$, namely the $*$-operation, $f^*(g) = \overline{f(g^{-1})}$, and the trace operation, $\mathrm{tr}(f) = f(1)$. Following [BHK, §4.1], $\mathcal{H}(G /\!/ K;\sigma)$ is a normalized Hilbert algebra with involution $f \mapsto f^*$ and scalar product

$$[f_1, f_2] = \frac{\mathrm{vol}(K)}{\dim \sigma}\,\mathrm{tr}(f_1^* f_2).$$

This structure yields a Plancherel formula on $\mathcal{H}(G /\!/ K;\sigma)$: there is a positive Borel measure μ_σ on the C^*-algebra completion $C^*(K,\sigma)$ of $\mathcal{H}(G /\!/ K;\sigma)$ such that

$$[f, 1_\sigma] = \int_{\widehat{C^*}(K,\sigma)} \mathrm{tr}\,\pi(f)\,\mathrm{d}\hat{\mu}_\sigma(\pi).$$

Note that μ_σ depends on the chosen Haar measure of G.

We now consider this situation for two groups, G_1, G_2. Fix an open compact subgroup $K_i \subseteq G_i$, an irreducible smooth representation σ_i of K_i, and a Haar measure μ_i of G_i. Let $\widehat{\mu}_i$ be the Plancherel measure on $\widehat{G}_i$ with respect to the Haar measure μ_i; following the notation of [BHK] we denote by ${}_r\widehat{G}_i$ the support of $\widehat{\mu}_i$. We write ${}_r\widehat{G}_i(\sigma_i)$ for the subspace of ${}_r\widehat{G}_i$ consisting of the representations π for which $(\pi \otimes \sigma_i^*)^{K_i} \neq 0$.

From [BHK, §5.2], if we have an isomorphism of Hecke algebras

$$\alpha : \mathcal{H}(G_1 /\!/ K_1;\sigma_1) \to \mathcal{H}(G_2 /\!/ K_2;\sigma_2)$$

such that

1. $\alpha(f^*) = \alpha(f)^*$, and
2. $\mathrm{tr}(f) = 0$ implies $\mathrm{tr}\big(\alpha(f)\big) = 0$,

for all $f \in \mathcal{H}(G_1 /\!/ K_1;\sigma_1)$, then it is an isomorphism of Hilbert algebras. In this situation, we can apply [BHK, Cor. C, p.57].

Lemma 1.2. *An isomorphism*

$$\alpha : \mathcal{H}(G_1 /\!/ K_1;\sigma_1) \to \mathcal{H}(G_2 /\!/ K_2;\sigma_2)$$

of Hilbert algebras induces a homeomorphsim

$$\widehat{\alpha} : {}_r\widehat{G}_2(\sigma_2) \to {}_r\widehat{G}_1(\sigma_1)$$

such that

$$\frac{\mu_1(K_1)}{\dim \sigma_1}\widehat{\mu}_1(\widehat{\alpha}(S)) = \frac{\mu_2(K_2)}{\dim \sigma_2}\widehat{\mu}_2(S)$$

for any Borel subset S *of* ${}_r\widehat{G}_2(\sigma_2)$.

In the latter sections, we will apply this lemma with $G_1 = \widetilde{\mathrm{Sp}}(W)$. Strictly speaking, the groups considered in [BHK] are connected, reductive k-groups; however, there is no obstruction in extending this result to the metaplectic group.

2. Hecke algebra isomorphisms

In this section, we define our Hecke algebras $H_\psi^\pm$ of $\widetilde{\mathrm{Sp}}(W)$ and show that they are isomorphic to the affine Hecke algebras $H^\pm$ of $\mathrm{SO}(V^\pm)$.

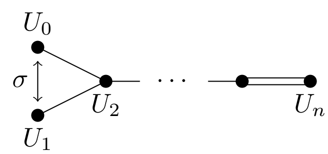
2.1. **Hecke algebra of** $\mathrm{SO}(V^+)$**.** Let V^+ be a quadratic space of dimension $2n+1$ with trivial discriminant and trivial Hasse invariant; then, $\mathrm{SO}(V^+)$ is a split, adjoint, orthogonal group of type B_n. Let I^+ and Ω_a^+ denote its Iwahori subgroup and affine Weyl group, respectively. The standard Iwahori-Hecke algebra is the set of smooth, compactly-supported I^+-bi-invariant functions on $\mathrm{SO}(V^+)$,

$$H^+ = \mathcal{H}(\mathrm{SO}(V^+) /\!\!/ I^+; \mathbf{1}).$$

For each $w \in \Omega_a^+$, take U_w to be the characteristic function on the double coset $I^+ w I^+$. The collection $\{U_w\}$ forms a basis of H^+ as a vector space. As an algebra, H^+ is generated by elements $U_0, \ldots, U_n$, and σ, where $U_i = U_{w_i}$ for w_i a simple affine reflection in Ω_a^+, and σ is the outer automorphism which exchanges the nodes on the Coxeter diagram corresponding to U_0 and U_1. The quadratic relations for the U_i are

$$(U_i + 1)(U_i - q) = 0,$$

and the braid relations are given by the affine diagram of type B_n.

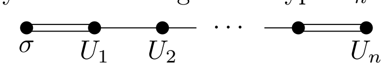


For details, see [IM, §3].

Noting that $\sigma^2 = 1$ and $\sigma U_1 \sigma = U_0$, we see that U_0 is abstractly unnecessary as a generator. Hence, H^+ is generated by σ, $U_1, \ldots, U_n$ subject to the quadratic relations,

$$(\sigma + 1)(\sigma - 1) = 0 \quad \text{and} \quad (U_i + 1)(U_i - q) = 0,$$

and the braid relations given by the affine diagram of type C_n.



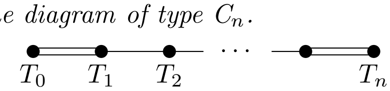
2.2. τ_0**-spherical Hecke algebra of** $\widetilde{\mathrm{Sp}}(W)$**.** The restriction of the minimal type τ_0 from $\widetilde{K}_0$ to the Iwahori subgroup $\widetilde{I}$ remains irreducible as proven in[SW]. In this section, we comput the τ_0-spherical Hecke algebra

$$H_\psi^+ = \mathcal{H}(\widetilde{\mathrm{Sp}}(W) /\!\!/ \widetilde{I}; \tau_0).$$

Theorem 2.1. *The Hecke algebra H_ψ^+ is generated by invertible elements $T_0, T_1, \ldots, T_n$, satisfying the quadratic relations*

$$(T_0 + 1)(T_0 - 1) = 0 \quad \textit{and} \quad (T_i + 1)(T_i - q) = 0 \quad \textit{for } i \neq 0,$$

and the braid relations of affine diagram of type C_n.



In particular, H_ψ^+ is abstractly isomorphic to H^+.

Furthermore, this isomorphism is an isomorphism of Hilbert algebras, and if the Haar measures on $\widetilde{\mathrm{Sp}}(W)$ and $\mathrm{SO}(V^+)$ are respectively normalized by

$$\mathrm{vol}(\widetilde{I}) = \dim(\tau_0) = q^{en} = |2|^{-n} \quad \textit{and} \quad \mathrm{vol}(I^+) = 1,$$

then the Plancherel measures on H^+_ψ and H^+ coincide.

Proof. We prove this theorem by investigating the structure of some 2-dimensional Hecke subalgebras. For $i = 0, \dots, n$, we define

$$\widetilde{I}_i = \widetilde{I} \cup \widetilde{I} s_i \widetilde{I},$$

and we take $H^+_{\psi,i}$ to be the subalgebra consisting of elements supported on $\widetilde{I}_i$; that is,

$$H^+_{\psi,i} = \mathcal{H}(\widetilde{I}_i /\!\!/ \widetilde{I}; \tau_0).$$

This subalgebra is at most 2-dimensional and is isomorphic to $\mathrm{End}_{\widetilde{I}_i}\big(\mathrm{Ind}_{\widetilde{I}}^{\widetilde{I}_i}(\tau_0^*)\big)$; it is exactly 2-dimensional if and only if the induced representation is reducible. Working in the dual setting, Frobenius reciprocity guarantees that the space $\tau_{0,i}$, generated by the action of $\widetilde{I}_i$ on τ_0, may be realized as a submodule of $\mathrm{Ind}_{\widetilde{I}}^{\widetilde{I}_i}(\tau_0)$, so it suffices to verify that it is a submodule of strictly smaller dimension. Since

$$d = \dim\big(\mathrm{Ind}_{\widetilde{I}}^{\widetilde{I}_i}(\tau_0)\big) = \dim(\tau_0) \cdot [\widetilde{I}_i : \widetilde{I}] = q^{en}(q+1),$$

and

$$\tau_{0,i} = \begin{cases} \tau_0 & \text{if } i \neq 0, \\ \tau_1^+ & \text{if } i = 0 \end{cases} \qquad \Longrightarrow \qquad d_1 = \dim(\tau_{0,i}) = \begin{cases} q^{en} & \text{if } i \neq 0, \\ \frac{1}{2} q^{en}(q+1) & \text{if } i = 0, \end{cases}$$

we see that $\mathrm{Ind}_{\widetilde{I}}^{\widetilde{I}_i}(\tau_0^*)$ is indeed reducible.

Therefore, there is an element T_i of $H^+_{\psi,i} \subset H^+_\psi$ which is supported precisely on $\widetilde{I} s_i \widetilde{I}$. In order to normalize T_i and to compute its quadratic relation, we consider the decomposition

$$\mathrm{Ind}_{\widetilde{I}}^{\widetilde{I}_i}(\tau_0^*) = \pi_1^* \oplus \pi_2^*,$$

where $\pi_1^* = \tau_{0,i}^*$ has dimension d_1 and π_2^* has dimension

$$d_2 = d - d_1 = \begin{cases} q^{en+1} & \text{if } i \neq 0, \\ \frac{1}{2} q^{en}(q+1) & \text{if } i = 0. \end{cases}$$

We normalize T_i to act by $\lambda_2 = -1$ on π_2^* and by λ_1 on π_1^*. Using Lemma 1.1, we have

$$\lambda_1 = \frac{d_2}{d_1} = \begin{cases} q & \text{if } i \neq 0, \\ 1 & \text{if } i = 0, \end{cases}$$

giving the desired quadratic relation $(T_i + 1)(T_i - \lambda_1) = 0$. The invertibility of T_i follows from its quadratic relation; explicitly,

$$T_0^{-1} = T_0 \quad \text{and} \quad T_i^{-1} = q^{-1}(T_i - q + 1) \quad \text{for } i \neq 0.$$

Suppose that we have a braid relation

$$s_i s_j \cdots = s_j s_i \cdots$$

in Ω_a. Then each of the Hecke algebra elements $T_i T_j \dots$ and $T_j T_i \dots$ is supported on the same $\widetilde{I}$-double coset. From the normalization of the T_i, each of these elements must act on $(\tau_0 \otimes \tau_0^*)^{\widetilde{I}}$ in the same way. Whence

$$T_i T_j \cdots = T_j T_i \cdots.$$

Therefore, the braid relations for the T_i are the same as those for the s_i, so any minimal expression $w = s_{i_1} \cdots s_{i_r}$ defines a Hecke algebra element $T_w = T_{s_1} \cdots T_{s_r}$ supported on $\widetilde{I} w \widetilde{I}$. From the quadratic and braid relations, we have an explicit isomorphism $H^+_\psi \to H^+$ given by

$$T_0 \mapsto \sigma \quad \text{and} \quad T_i \mapsto U_i \quad \text{for } i \neq 0.$$

Next let us show that this isomorphism is an isomorphism of Hilbert algebras. As each Hecke algebra is supported on its respective affine Weyl group, we have that

$$\operatorname{tr}(T_w) = \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{if } w \neq 1 \end{cases} \qquad \text{and} \qquad \operatorname{tr}(U_w) = \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{if } w \neq 1, \end{cases}$$

so the trace-zero property is clearly preserved. For $w \in \Omega_a^+$, the I^+-double cosets of w and w^{-1} are equal, so the $*$-operation in H^+ satisfies $U_i^* = U_i$, and hence, $U_w^* = U_{w^{-1}}$. In H_ψ^+, we have that T_i^* and T_i are both supported on $\widetilde{I}s_i\widetilde{I}$, so T_i^* acts on τ_0^* by a constant. For $\phi \in \tau_0^*$, $[\phi, T_i^*\phi] = [T_i\phi, \phi]$, so T_i^* and T_i must act by the same constant. Thus, $T_i^* = T_i$ and $T_w^* = T_{w^{-1}}$, so the $*$-operation is also preserved. Therefore, the isomorphism $H_\psi^+ \cong H^+$ is an isomorphism of Hilbert algebras.

From the normalization given in the statement of the theorem, the preservation of the Plancherel measures follows immediately from Lemma 1.2. □

Corollary 2.2. *The isomorphism $H_\psi^+ \cong H^+$ preserves the formal degree of the Steinberg representations of the respective Hecke algebras.*

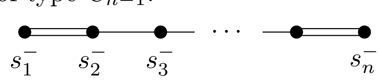
Remark. If $p \neq 2$, then the proof for the isomorphism $H_\psi^+ \cong H^+$ is essentially the one given in [GS2]. One notable difference is in the specific normalization of Hecke operators, which is always a delicate issue. In [GS2], they work in the central extension $\widetilde{\mathrm{Sp}}(W)_8$ of $\mathrm{Sp}(W)$ by the 8th roots of unity and normalize the generating Hecke operators to act on certain lifts of affine reflections in a specified way. We have opted to normalize the generating Hecke operator T_i to act by -1 on the irreducible representation not containing the minimal type τ_0^*.

The other notable difference is the implementation of the theory of induced Plancherel measures from [BHK]. In [GS2], they show directly that the formal degrees of the respective Steinberg representations coincide. This computation is avoided here because it follows from the more general coincidence of the induced Plancherel measures.

Remark. Assuming $k = \mathbb{Q}_2$, an isomorphic Hecke algebra is constructed in [W] by finding a 1-dimensional type for a subgroup of $\widetilde{I}$. This construction extends to the case where k is an unramified extension of $\mathbb{Q}_2$ but does not appear to work for ramified extensions.

Remark. A change in conductor ψ will yield an identical isomorphism of Hecke algebras using essentially the same arguments. If the conductor is even, the minimal type τ_0 is isomorphic to the one used here, so the computation of the Hecke algebra H_ψ^+ will be virtually identical. If the conductor is odd, the minimal type τ_n is isomorphic to the one employed here, so the computation of the Hecke algebra will require some modification, primarily in notation.

2.3. **Hecke algebra of** $\mathrm{SO}(V^-)$**.** For the remainder of the section, we suppose that $n \geq 2$. Let V^- be a quadratic space of dimension $2n+1$ with trivial discriminant and non-trivial Hasse invariant; then, $\mathrm{SO}(V^-)$ is the non-split inner form of $\mathrm{SO}(V^+)$. Let I^- be the Iwahori subgroup of $\mathrm{SO}(V^-)$, which is the pointwise stabilizer of a fundamental chamber in its Bruhat-Tits building, and Ω_a^- its affine Weyl group, which is generated by reflections $s_1^-, \ldots, s_n^-$ subject to the braid relations of the affine diagram of type C_{n-1}.



The standard Iwahori-Hecke algebra is the set of smooth, compactly-supported I^--bi-invariant functions on $\mathrm{SO}(V^-)$,

$$H^- = \mathcal{H}(\mathrm{SO}(V^-) /\!\!/ I^-; \mathbf{1}).$$

For each $w \in \Omega_a^-$, let U_w be the characteristic function on the double coset $I^- w I^-$. The collection $\{U_w\}$ forms a basis of H^- as a vector space. As an algebra, H^- is generated by $U_1, \ldots, U_n$, where $U_i = U_{s_i^-}$. These generators satisfy the quadratic relations,

$$(U_1 + 1)(U_1 - q^2) = 0 \quad \text{and} \quad (U_i + 1)(U_i - q) = 0 \quad \text{for } i \neq 1,$$

and the same braid relations as the s_i^-. See [GS2] or [Ti] for details.

2.4. **τ_1^--spherical Hecke algebra of $\widetilde{\mathrm{Sp}}(W)$.** We define the open compact subgroup $\widetilde{J} \subseteq \widetilde{\mathrm{Sp}}(W)$ to be the full inverse image of

$$J = \bigcap_{i=1}^{n} K_i = I \cup I w_0 I,$$

and consider the restriction of τ_1^- to $\widetilde{J}$. The group $\widetilde{J}$ contains the metaplectic preimage of the subgroup

$$\Phi_{-2\epsilon_1+1}\big(\mathrm{SL}_2(\mathcal{O})\big) \times I_{n-1},$$

where I_{n-1} is an Iwahori subgroup of the symplectic group of type C_{n-1}. From [SW], each component of this direct product acts irreducibly on the corresponding component of the tensor product

$$\tau_1^- = S(\mathcal{O}/2\varpi\mathcal{O})^- \otimes S(\mathcal{O}^{n-1}/2\mathcal{O}^{n-1}),$$

hence the restriction of τ_1^- to $\widetilde{J}$ must remain irreducible. In this section, we compute the τ_1^--spherical Hecke algebra

$$H_\psi^- = \mathcal{H}(\widetilde{\mathrm{Sp}}(W) /\!\!/ \widetilde{J}; \tau_1^-).$$

We define $\Omega_a' = \langle s_1', \ldots, s_n' \rangle \subseteq \Omega_a$, where

$$s_i' = \begin{cases} s_i & \text{if } i \neq 1 \\ s_1 s_0 s_1 & \text{if } i = 1. \end{cases}$$

The reflection s_1' corresponds to the affine reflection s_{-2e_2+1}, hence Ω_a' is isomorphic to the affine Weyl group of type C_{n-1}, i.e., to Ω_a^-. We have the following lemma, the proof of which is only a slight modification of the proof in [GS2, Lemma 10].

Lemma 2.3. *The support of H_ψ^- is contained in $\widetilde{J}\Omega_a'\widetilde{J}$.*

Proof. Fix $f \in H_\psi^-$ and $w \in \Omega_a$. Write $w = a\sigma$, where $\sigma \in \Omega$ and a is translation by $(a_1, \ldots, a_n)$. As $J = I \cup I s_0 I$, we have that w and $s_0 w$ represent the same $\widetilde{J}$-double coset, so it suffices to show that $f(w) \neq 0$ implies that either w or $s_0 w$ is in Ω_a'.

We recall two facts about the Weil representation on $S(Y)$:

1. for all $\alpha \in \Sigma$, we have $\widetilde{x}_{-\alpha}(u)\phi = \phi$ if and only if $\widetilde{x}_\alpha(-u)\widehat{\phi} = \widehat{\phi}$;
2. for $\alpha = 2\epsilon_j$, we have $\widetilde{x}_\alpha(u)\phi(y) = \psi(u y_j^2)\phi(y)$.

Together, these give us the following criteria:

$$\widetilde{x}_\alpha(u) \in \ker \tau_1^- \quad \text{if and only if} \quad \begin{cases} u \in 4\mathcal{O} & \text{if } \alpha \neq -2\epsilon_1, \\ u \in 4\varpi^2\mathcal{O} & \text{if } \alpha = -2\epsilon_1. \end{cases}$$

The element w conjugates the root group of α_0 to the root group of $\beta + m$, where

$$\beta + m = w(\alpha_0) = \sigma^{-1}(-2\epsilon_1) + (1 - 2a_1).$$

Hence, w or $s_0 w$ is in Ω_a' if and only if $\beta + m \neq \pm\alpha_0$.

Let $t \in 4\mathcal{O}$ be such that

$$w^{-1}\widetilde{x}_{\alpha_0}(4)w = \widetilde{x}_{\beta+m}(t).$$

First suppose that $m > -1$ and $\beta \neq -2\epsilon_1$ or that $m > 1$ and $\beta = -2\epsilon_1$. Then, $\widetilde{x}_{\beta+m}(t) \in \ker \tau_1^-$ and $\widetilde{x}_{\alpha_0}(4) \notin \ker \tau_1^-$, hence

$$f(w) = f(w)(\tau_1^-)^*\big(\widetilde{x}_{\beta+m}(t)\big) = f\big(w\widetilde{x}_{\beta+m}(t)\big) = f\big(\widetilde{x}_{\alpha_0}(4)w\big) = (\tau_1^-)^*\big(\widetilde{x}_{\alpha_0}(4)\big)f(w),$$

giving that $f(w) = 0$.

Now suppose that $m < 1$ and $\beta \neq 2\epsilon_1$ or that $m < -1$ and $\beta = 2\epsilon_1$. Then, $\widetilde{x}_{-\beta-m}(t) \in \ker \tau_1^-$ and $\widetilde{x}_{-\alpha_0}(4) \notin \ker \tau_1^-$, hence

$$f(w) = f(w)(\tau_1^-)^*\big(\widetilde{x}_{-\beta-m}(t)\big) = f\big(w\widetilde{x}_{-\beta-m}(t)\big) = f\big(\widetilde{x}_{-\alpha_0}(4)w\big) = (\tau_1^-)^*\big(\widetilde{x}_{-\alpha_0}(4)\big)f(w),$$

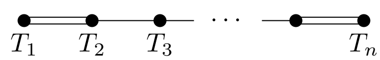
giving that $f(w) = 0$.

In sum, if $f(w) \neq 0$, then $\beta + m$ must equal $\pm\alpha_0$, and the lemma is proved. □

Theorem 2.4. *The Hecke algebra H_ψ^- is generated by invertible elements $T_1, \ldots, T_n$, satisfying the quadratic relations*

$$(T_1+1)(T_1-q^2) = 0 \quad \textit{and} \quad (T_n+1)(T_n-q) = 0 \quad \textit{for } i \neq 0,$$

and the braid relations of the affine diagram of type C_{n-1}.



In particular, H_ψ^- is abstractly isomorphic to H^-.

Furthermore, this isomorphism is an isomorphism of Hilbert algebras and if the Haar measures on $\widetilde{\mathrm{Sp}}(W)$ and $\mathrm{SO}(V^-)$ are respectively normalized by

$$\mathrm{vol}(\widetilde{J}) = \dim \tau_1^- = \tfrac{1}{2}q^{en}(q-1) \quad \textit{and} \quad \mathrm{vol}(I^-) = 1,$$

then the Plancherel measures on H_ψ^- and H^- coincide.

Proof. This proof is similar to the proof of Theorem 2.1. We investigate the structure of some 2-dimensional Hecke subalgebras in order to see that H_ψ^- is supported exactly on $\widetilde{J}\Omega_a'\widetilde{J}$. For $i = 1, \ldots, n$, we define $\widetilde{J}_i$ to be the group generated by $\widetilde{J}$ and $\widetilde{J}s_i'\widetilde{J}$.

If $i \neq 1$, then $\widetilde{J}_i = \widetilde{J} \cup \widetilde{J}s_i\widetilde{J}$, hence $[\widetilde{J}_i : \widetilde{J}] = q+1$; for the case $i = 1$, we note that $\widetilde{J}_1$ is the union of those $\widetilde{I}w\widetilde{I}$ for which w is in the group generated by s_0 and s_1, hence

$$[\widetilde{J}_1 : \widetilde{J}] = \frac{[\widetilde{J}_1 : \widetilde{I}]}{[\widetilde{J} : \widetilde{I}]} = \frac{1+2q+2q^2+2q^3+q^4}{1+q} = 1+q+q^2+q^3.$$

We take $H_{\psi,i}^-$ to be the subalgebra of H_ψ^- consisting of elements supported on $\widetilde{J}_i$; that is,

$$H_{\psi,i}^-\mathcal{H}(\widetilde{J}_i /\!\!/ \widetilde{J}; \tau_1^-).$$

This subalgebra is at most 2-dimensional and is isomorphic to $\mathrm{End}_{\widetilde{J}_i}\left(\mathrm{Ind}_{\widetilde{J}}^{\widetilde{J}_i}(\tau_1^-)^*\right)$.

Let $\tau_{1,i}^-$ be the space generated by the action of $\widetilde{J}_i$ on τ_1^-. We note that if $i \neq 1$, then $\tau_{1,i}^- = \tau_1^-$ since each of $s_2, \ldots, s_n$ preserves the lattice associated to τ_1^-. However, since $\widetilde{J}_1$ includes $\widetilde{I}s_1\widetilde{I}$ and since s_1 acts essentially by permuting the first two coordinates of $S(Y)$, we have that $\tau_{1,1}^- = \tau_2^-$.

To see that $H_{\psi,i}^-$ is exactly 2-dimensional, we work in the dual setting and note that

$$d_1 = \dim \tau_{1,i}^- = \begin{cases} \dim(\tau_1^-) & \text{if } i \neq 1, \\ \dim(\tau_2^-) & \text{if } i = 1 \end{cases} = \begin{cases} \frac{1}{2}q^{en}(q-1) & \text{if } i \neq 1, \\ \frac{1}{2}q^{en}(q^2-1) & \text{if } i = 1 \end{cases}$$

is strictly smaller than

$$d = \dim\left(\operatorname{Ind}_{\widetilde{J}}^{\widetilde{J}_i}(\tau_1^-)\right) = \dim(\tau_1^-)\cdot[\widetilde{J}_i : \widetilde{J}] = \begin{cases} \frac{1}{2}q^{en}(q^2-1) & \text{if } i \neq 1, \\ \frac{1}{2}q^{en}(q^4-1) & \text{if } i = 1. \end{cases}$$

Therefore, for $i = 1, \dots, n$, there exists $T_i \in H_\psi^-$ supported precisely on $\widetilde{J}s_i\widetilde{J}$. We consider the decomposition

$$\operatorname{Ind}_{\widetilde{J}}^{\widetilde{J}_i}(\tau_1^-)^* = \pi_1^* \oplus \pi_2^*,$$

where $\pi_1^* = (\tau_{1,i}^-)^*$ has dimension d_1 and π_2^* has dimension

$$d_2 = d - d_1 = \begin{cases} \frac{1}{2}q^{en}(q^2-q) & \text{if } i \neq 1, \\ \frac{1}{2}q^{en}(q^4-q^2) & \text{if } i = 1. \end{cases}$$

We normalize T_i to act by $\lambda_2 = -1$ on π_2^* and by λ_1 on π_1^*. Using Lemma 1.1, we have

$$\lambda_1 = \frac{d_2}{d_1} = \begin{cases} q & \text{if } i \neq 1, \\ q^2 & \text{if } i = 1, \end{cases}$$

giving the desired quadratic relation $(T_i+1)(T_i-\lambda_1) = 0$. The invertibility of T_i follows from its quadratic relation; explicitly, $T_i^{-1} = \lambda_1^{-1}(T - \lambda_1 + 1)$.

The proof of the braid relations mimics the proof of Theorem 2.1 with $\widetilde{J}$ instead of $\widetilde{I}$, τ_1^- instead of τ_0, and Ω_a' instead of Ω_a. We note that computations involving $\widetilde{J}$-double cosets involve a weighted length function ℓ' on Ω_a', defined by setting $\ell'(s_1') = 3$ and $\ell'(s_i') = 1$ if $i \neq 1$. The details of this length function are contained in [GS2, Prop. 1]; it suffices to mention here that

1. $[JwJ : J] = q^{\ell_3'(w)}$ for all $w \in \Omega_a'$.
2. If $w_1, w_2 \in \Omega_a'$ satisfy $\ell_3'(w_1) + \ell_3'(w_2) = \ell_3'(w_1w_2)$, then $Jw_1J \cdot Jw_2J = Jw_1w_2J$.

For a minimal expression $w = s_{i_1}' \cdots s_{i_r}'$ in Ω_a', the braid relations in H_ψ^- allow us to define a canonical Hecke operator $T_w = T_{i_1} \cdots T_{i_r}$ supported precisely on the double coset $\widetilde{J}w\widetilde{J}$. From the quadratic and braid relations, we have the explicit isomorphism $H_\psi^- \to H^-$ given by $T_i \mapsto U_i$.

As in the proof of Theorem 2.1, one can show that $H_\psi^- \cong H^-$ as Hilbert algebras, hence Lemma 1.2 will give the coincidence of Plancherel measures. $\square$

Corollary 2.5. *The isomorphism $H_\psi^- \cong H^-$ preserves the formal degree of the Steinberg representations of the respective Hecke algebras.*

Remark. A change in conductor will yield an identical Hecke algebra isomorphism. If the conductor is even, the minimal type τ_1^- is isomorphic to the one employed here and the computation of the Hecke algebra would go through essentially unchanged. If the conductor is odd, the minimal type τ_{n-1}^- is isomorphic to the one used here, so some modification would be required in the computation of H_ψ^-. The reader is referred to [GS2], where Gan and Savin use an odd conductor in their computation of H_ψ^- under the assumption that $p \neq 2$.

3. Equivalence of categories between $\mathcal{G}_\psi^\pm$ and $\mathcal{S}_0^\pm$

In the category of smooth genuine representations of $\widetilde{\mathrm{Sp}}(W)$, let $\mathcal{G}_\psi^\pm$ be the component containing the even/odd Weil representation $\omega_\psi^\pm$. In the category of smooth representations of $\mathrm{SO}(V^\pm)$, let $\mathcal{S}_0^\pm$ be the component containing the trivial representation.

We will prove our main theorem, namely that there is an equivalence of categories between $\mathcal{G}_\psi^+$ and $\mathcal{S}_0^+$ and an equivalence of categories between $\mathcal{G}_\psi^-$ and $\mathcal{S}_0^-$. Our proof essentially follows that of [GS2].

3.1. **Equivalence between $\mathcal{G}_\psi^+$ and $\mathcal{S}_0^+$.** Let U (resp. U^-) be the unipotent radical in $\mathrm{Sp}(W)$ generated by positive (resp. negative) root groups. Let $\widetilde{B} = \widetilde{T}U \subseteq \widetilde{\mathrm{Sp}}(W)$ be the preimage of the Borel subgroup $B = TU$ of $\mathrm{Sp}(W)$. (Recall that the unipotent radical U splits in $\widetilde{\mathrm{Sp}}(W)$.)

An element t of the maximal torus T may be expressed uniquely as

$$t = (t_1, \dots, t_n) = h_{2\epsilon_1}(t_1) \cdots h_{2\epsilon_n}(t_n),$$

hence we have a canonical lift of t given by

$$\widetilde{t} = \widetilde{h}_{2\epsilon_1}(t_1) \cdots \widetilde{h}_{2\epsilon_n}(t_n).$$

With this convention, multiplication in $\widetilde{T}$ is given by

$$\widetilde{t} \cdot \widetilde{u} = (t, u)\, \widetilde{tu} = \prod_{i=1}^{n} (t_i, u_i)\, \widetilde{h}_{2\epsilon_i}(t_i u_i),$$

where the cocycle $(t, u) \in \{\pm 1\}$ is the product of Hilbert symbols (t_i, u_i) on k.

Recalling the action of T on Y by $ty = (t_1 y_1, \dots, t_n y_n)$, the ω_ψ-action of $\widetilde{t}$ on $S(Y)$ is given by

$$\widetilde{t}\phi(y) = \beta_t |\det t|^{1/2} \phi(ty),$$

where β_t is a 4th root of unity satisfying $\beta_t \beta_u = (t, u)\beta_{tu}$.

Given a character $\chi = (\chi_1, \dots, \chi_n)$ on T, we define a genuine character on $\widetilde{T}$ by

$$\widetilde{\chi}(\widetilde{t}) = \chi(t)\beta_t.$$

We extend this character trivially to all of $\widetilde{B}$ and define $I(\widetilde{\chi})$ to be the normalized induced representation $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{Sp}}(W)} \widetilde{\chi}$. By Frobenius reciprocity,

$$\mathrm{Hom}_{\widetilde{\mathrm{Sp}}(W)}(\pi, I(\widetilde{\chi})) \cong \mathrm{Hom}_{\widetilde{T}}(\pi_U, \widetilde{\chi}), \tag{3.1}$$

where π is any smooth representation of $\widetilde{\mathrm{Sp}}(W)$ and π_U is the normalized Jacquet module with respect to the Borel $\widetilde{B}$.

Lemma 3.2. *The component $\mathcal{G}_\psi^+$ is precisely the component whose irreducible representations are submodules of $I(\widetilde{\chi})$ for some unramified character χ.*

Proof. The functional $l : S(Y)^+ \to \mathbb{C}$ defined by $l(\phi) = \phi(0)$ factors through the Jacquet module $(\omega_\psi^+)_U$ and gives a non-trivial element in $\mathrm{Hom}_{\widetilde{T}}((\omega_\psi^+)_U, \widetilde{\chi})$ for some unramified χ, which gives an embedding $\omega_\psi^+ \subseteq I(\widetilde{\chi})$ via Frobenius reciprocity. □

The Iwahori subgroup $\widetilde{I}$ admits a factorizatioin

$$\widetilde{I} = \widetilde{I}_{U^-} \widetilde{I}_T \widetilde{I}_U,$$

where $\widetilde{I}_{U^-} = \widetilde{I} \cap U^-$, $\widetilde{I}_T = \widetilde{I} \cap \widetilde{T}$, and $\widetilde{I}_U = \widetilde{I} \cap U$.

Now let us define the "Jacquet module" $(\tau_0)_U$ of τ_0 with respect to $\widetilde{I}_U$; that is, $(\tau_0)_U$ is the quotient of the $S(L_0/2L_0)$ by

$$\langle \tau_0(u)\phi - \phi : u \in \widetilde{I}_U, \phi \in S(L_0/2L_0) \rangle,$$

which may be viewed as a representation of $\widetilde{I}_T$.

Lemma 3.3. *The space $(\tau_0)_U$ is one dimensional and spanned by the image of the characteristic function of $2L_0$. Moreover each element $\widetilde{t} \in \widetilde{I}_T$ acts by β_t on $(\tau_0)_U$.*

Proof. Suppose that $\phi \in \tau_0$ is supported on $a + 2L_0$ for $a \in L_0 \smallsetminus 2L_0$ and let i be such that $a_i \in \mathcal{O}^\times$. The element $\widetilde{x}_{2\epsilon_i}(1)$ acts on ϕ by the constant $\psi(a_i^2) \neq 1$. Therefore, the image of ϕ in $(\tau_0)_U$ is trivial.

On the other hand, if ϕ is the characteristic function on $2L_0$, then $\widetilde{I}_U$ acts trivially on ϕ and $\widetilde{t} \in \widetilde{I}_T$ acts by $\widetilde{t}\phi(y) = \beta_t\phi(ty) = \beta_t\phi(y)$. □

Theorem 3.4. *The functor from the category $\mathcal{G}_\psi^+$ to the category of H_ψ^+-modules, given by*

$$\pi \mapsto (\pi \otimes \tau_0^*)^{\widetilde{I}},$$

is an equivalence of categories. In particular, there is an equivalence of categories between $\mathcal{G}_\psi^+$ and $\mathcal{S}_0^+$ given by the isomorphism $H_\psi^+ \cong H^+$ of Hecke algebras. Furthermore, this equivalence preserves the temperedness and square integrability of representations.

Proof. We have the natural surjection

$$r : (\pi \otimes \tau_0^*)^{\widetilde{I}} \to (\pi_U \otimes (\tau_0^*)_U)^{\widetilde{I}_T},$$

which is a slight variant of what is called "Jacquet's Lemma" in [B2, 64-65]. (To prove our version, one can follow the argument there. Also, see [B1, Prop. 3.5.2].)

We first show that r is an isomorphism. Suppose that $v \in \ker r$, i.e., that there exists an open compact subgroup U_v of U such that $\int_{U_v} \pi(u)v\,du = 0$.

For a translation $\lambda = (\lambda_1, \ldots, \lambda_n)$ in $D \subseteq \Omega_a$, write

$$\lambda = \widetilde{h}_{2\epsilon_1}(\varpi^{\lambda_1}) \cdots \widetilde{h}_{2\epsilon_n}(\varpi^{\lambda_n})$$

as its representative in $\widetilde{T}$. Take $\lambda \in D$ such that $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ and $\lambda^{-1}\widetilde{I}_U\lambda \supseteq U_v$. Then,

$$\widetilde{I}\lambda\widetilde{I} = \bigcup_{i=1}^{q^{\ell(\lambda)}} \lambda u_i \widetilde{I}$$

where the u_i are representatives of the $\widetilde{I}_U$-cosets in $\lambda^{-1}\widetilde{I}_U\lambda$. Let T_λ be the Hecke algebra element supported on $\widetilde{I}\lambda\widetilde{I}$ obtained using a minimal expression for $\lambda \in \Omega_a$ as in Theorem 2.1. Then,

$$\pi(T_\lambda)v = \pi(\lambda)\sum_{i=1}^{q^{\ell(\lambda)}} \pi(u_i)v = \pi(\lambda)\int_{\lambda^{-1}\widetilde{I}_U\lambda} \pi(u)v\,du = 0.$$

The element T_λ is invertible, as it is the product of invertible elements, hence $v = 0$ and r is injective.

Let π be an irreducible representation of $\widetilde{\mathrm{Sp}}(W)$ such that $(\pi \otimes \tau_0^*)^{\widetilde{I}} \neq 0$. As r is an isomorphism, $(\pi_U \otimes (\tau_0^*)_U)^{\widetilde{I}_T} \neq 0$. This implies that $\mathrm{Hom}_{\widetilde{T}}(\pi_U, \widetilde{\chi}) \neq 0$ for some unramified χ, since $\widetilde{t} \in \widetilde{I}_T$ acts by β_t on $(\tau_0)_U$. Therefore, by Frobenius reciprocity, we have that π is a subrepresentation of $I(\widetilde{\chi})$.

Conversely, let π be an irreducible submodule of $I(\widetilde{\chi})$ for some unramified χ. By Frobenius reciprocity, we have that

$$0 \neq (\pi_U \otimes (\tau_0^*)_U)^{\widetilde{I}_T} \cong (\pi \otimes \tau_0^*)^{\widetilde{I}}.$$

Thus, condition (iii) of [BK, 3.11] is satisified, which proves the equivalence of categories.

To complete the proof, we note that, as categories,

$$\mathcal{S}_0^+ \cong H^+\text{-modules} \cong H_\psi^+\text{-modules} \cong \mathcal{G}_\psi^+;$$

moreover, the trivial representation of $\mathrm{SO}(V^+)$ corresponds to the trivial module of $H^+ \cong H^+_\psi$, and hence to the even Weil represntation ω^+_ψ.

Under this equivalence, the preservation of the temperedness and square integrability of representations follows from Lemma 1.2. □

Remark. In [GS2], the preservation of the temperedness and square integrability is shown by using Casselman's criterion. In this paper, it is an immediate corollary of Lemma 1.2, once the Hecke algebra isomorphism $H^+_\psi \cong H^+$ is shown to be an isomorphism of Hilbert algebras.

3.2. **Equivalence between $\mathcal{G}^-_\psi$ and $\mathcal{S}^-_0$.** Consider the partial flag

$$X_n \subseteq X_{n-1} \subseteq \cdots \subseteq X_2$$

where X_i is the k-span of $\mathbf{e}_i, \ldots, \mathbf{e}_n$. Let $P = MN$ be the parabolic subgroup which is the stabilizer of this partial flag. Let W_1 be the symplectic subspace spanned by $\{\mathbf{e}_1, \mathbf{f}_1\}$ so that $\mathrm{Sp}(W_1) = \mathrm{SL}_2(k)$. Define $\omega_{\psi,1}$ to be the Weil representation of $\widetilde{\mathrm{Sp}}(W_1)$, realized as a representation in the space $S(k\mathbf{f}_1)$. This representation decomposes into even and odd parts; the odd part $\omega^-_{\psi,1}$ is supercuspidal.

Let $\widetilde{P} = \widetilde{M}N$ be the preimage of P in $\widetilde{\mathrm{Sp}}(W)$. Each element $m \in \widetilde{M}$ is uniquely written as

$$m = m_1 \cdot \widetilde{h}_{2\epsilon_2}(t_2) \cdots \widetilde{h}_{2\epsilon_n}(t_n)$$

where $t_i \in k^\times$ and $m_1 \in \widetilde{\mathrm{Sp}}(W_1)$. For each tuple of characters $\chi = (\chi_2, \ldots, \chi_n)$, we define a genuine representation $\omega^-_{\psi,1} \otimes \chi$ of $\widetilde{M}$ in the obvious way. We set

$$I(\omega^-_{\psi,1} \otimes \chi) = \mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(W)}(\omega^-_{\psi,1} \otimes \chi)$$

to be the normalized induced representation. For a smooth representation π of $\widetilde{\mathrm{Sp}}(W)$ and π_N its normalized Jacque module, we have Frobenius reciprocity:

$$\mathrm{Hom}_{\widetilde{\mathrm{Sp}}(W)}(\pi, I(\omega^-_{\psi,1} \otimes \chi)) \cong \mathrm{Hom}_{\widetilde{M}}(\pi_N, \omega^-_{\psi,1} \otimes \chi). \tag{3.5}$$

Lemma 3.6. *The component $\mathcal{G}^-_\psi$ is precisely the component whose irreducible representations are submodules of $I(\omega^-_{\psi,1} \otimes \chi)$ for an unramified character χ.*

Proof. The functional $l : S(Y)^- \to S(k\mathbf{f}_1)^-$ defined by restriction of functions from Y to $k\mathbf{f}_1$ factors through the Jacquet module $(\omega^-_\psi)_N$. Therefore, there is a non-trivial element in $\mathrm{Hom}_{\widetilde{M}}((\omega^-_\psi)_N, \omega^-_{\psi,1} \otimes \chi)$ for some unramified χ, which gives an embedding $\omega^-_\psi \subseteq I(\chi)$ via Frobenius reciprocity. □

Theorem 3.7. *The functor from the category $\mathcal{G}^-_\psi$ to the category of H^-_ψ-modules, given by*

$$\pi \mapsto (\pi \otimes (\tau^-_1)^*)^{\widetilde{J}},$$

is an equivalence of categories. In particular, there is an equivalence of categories between $\mathcal{G}^-_\psi$ and $\mathcal{S}^-_0$ given by the isomorphism $H^-_\psi \cong H^-$ of Hecke algebras. Furthermore, this equivalence preserves the temperedness and square integrability of representations.

Proof. Let $\widetilde{J}_M = \widetilde{J} \cap \widetilde{M}$ and $\widetilde{J}_N = \widetilde{J} \cap N$. As in the previous subsection, we define the "Jacquet module" $(\tau^-_1)_N$ with respect to $\widetilde{J}_N$, which we view as a representation of $\widetilde{J}_M$. Recall that

$$\tau^-_1 = S(\mathcal{O}/2\varpi\mathcal{O})^- \otimes S(\mathcal{O}^{n-1}/2\varpi\mathcal{O}^{n-1}),$$

and hence,

$$(\tau^-_1)_N = S(\mathcal{O}/2\varpi\mathcal{O})^- \otimes (\tau_{0,n-1})_{U_{n-1}},$$

where the second factor is the Jacquet module from the previous subsection in rank $n-1$. Therefore, $(\tau_1^-)_N$ is an irreducible representation of $\widetilde{J}_M$.

We have the natural surjection

$$r : (\pi \otimes (\tau_1^-)^*)^{\widetilde{J}} \to (\pi_N \otimes (\tau_1^-)^*_N)^{\widetilde{J}_M}.$$

Just as in Proposition 3.7, one can show that r is injective, which together with Frobenius reciprocity shows that $(\pi \otimes (\tau_1^-)^*)^{\widetilde{J}} \neq 0$ if and only if π is a submodule of $I(\omega_{\psi,1}^- \otimes \chi)$ for some unramified χ. Hence [BK, (3.11)] implies the equivalence of the categories.

As before, the preservation of the temperedness and square integrability of representations follows from Lemma 1.2. □

Remark. As a final remark, let us mention that in [GS2, Sec. 15 and 16] it is shown that the theta correspondence preserves unramified Langlands parameters, which relies on another work [GS1] of Gan and Savin. The only obstruction to remove the $p \neq 2$ assumption from [GS1], however, is the Howe duality conjecture, which was recently proven by Gan and the first-named author in [GT] for the case at hand. Hence, everything discussed in [GS2, Sec. 15 and 16] holds without the assumption $p \neq 2$.

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